FIN 411 -- Investments
Option Pricing

Simple arbitrage relations

Payoffs to call options

Black-Scholes model

Put-Call Parity

Implied Volatility

Options: Definitions

A call option gives the buyer the right, but not the obligation,

- to purchase a specific asset
- for a prespecified price
- on a specific future date

American vs. European Options

- exercisable before maturity?

Arbitrage Restrictions on Call Prices

1) $C \geqslant 0$

Consider the portfolio formed by buying the call option.

Today	Later: $S^* \leqslant K$	Later: $S^* > K$
$-C$	0	$S^* - K > 0$

Arbitrage Restrictions on Call Prices

2) $C \leqslant S$

Consider the following portfolio:
Buy the stock and sell the call.

Today	Later: $S^* \leqslant K$	Later: $S^* > K$
$C - S$	S^*	$S^* - (S^* - K) = K$

Arbitrage Restrictions on Call Prices

3) $C \geqslant S - PV(K)$

Consider the following portfolio:
Buy the call, sell the stock, and lend PV(K).

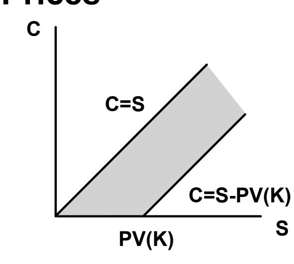
Today	Later: $S^* \leqslant K$	Later: $S^* > K$
$S - C - PV(K)$	$K - S^*$ >0	$(S^* - K) - S^* + K$ $=0$

Arbitrage Restrictions on Call Prices

1) $C \geqslant 0$

2) $C \leqslant S$

3) $C \geqslant S - PV(K)$



Early Exercise of American Options

Suppose you own a call option and you want to close out your position.

- You can exercise and receive S - K
- Or you can sell your option for its current market price C
- You choose the alternative that yields the greatest profit (e.g., exercise if C < S - K and sell if C > S - K)

Arbitrage Restrictions on American Call Prices

Suppose $C \leq S-K$ between ex-dividend days.
Then buy 1 call, short stock, lend K
close out position just before ex-dividend day

Today	At Ex-dividend Day	
	$S^* < K$	$S^* > K$
$-C+S-K \geq 0$	$-S^*+(1+r)K>0$	$(S^*-K)-S^*+(1+r)K = r K > 0$

Arbitrage Restrictions on American Call Prices

C > S - K except at expiration or just prior to an ex-dividend day

It is never optimal to exercise an American Call Option except at expiration or possibly just before an ex-dividend day.

A Call Option is "worth more alive than dead"

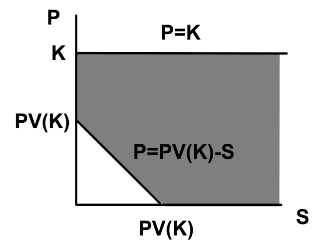
Arbitrage Restrictions on Put Prices

1) $P \geq 0$

2) $P \leq K$

3) $P \geq PV(K) - S$

4) $P \geq K - S$
(American Put Only)

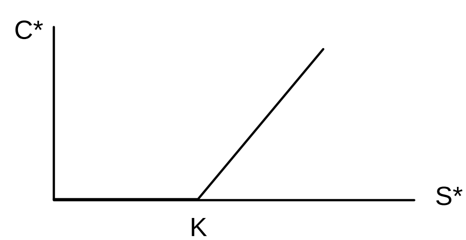


Payoffs Diagrams for Contingent Claims

Shows relation between $Payoff and Stock Price for claims with an exercise price, K = $100

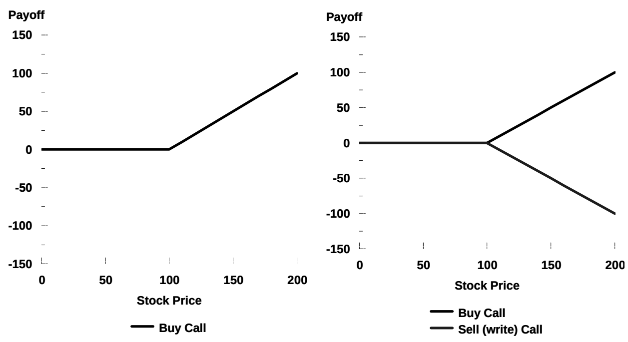
- ignores cost of buying the contingent claims/options
- ignores transaction costs
- useful for seeing relations among different contracts

Options: Payoff Diagrams

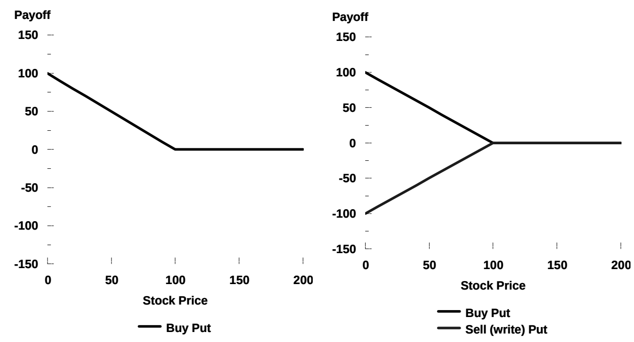


$C^* = \max [(S^* - K) , 0)]$
C^*= call at maturity t=T
S^*= stock at maturity
K= exercise price

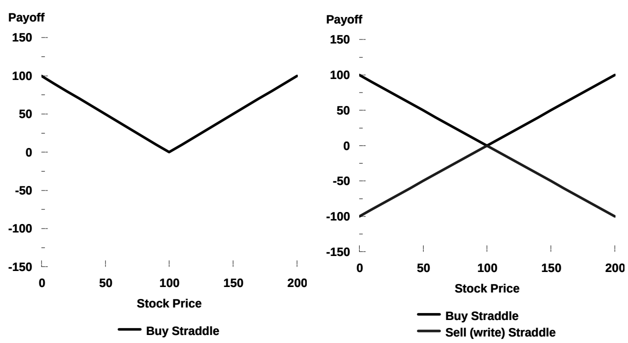
Payoffs to Buying & Selling Call Options [Exercise Price, K=$100]



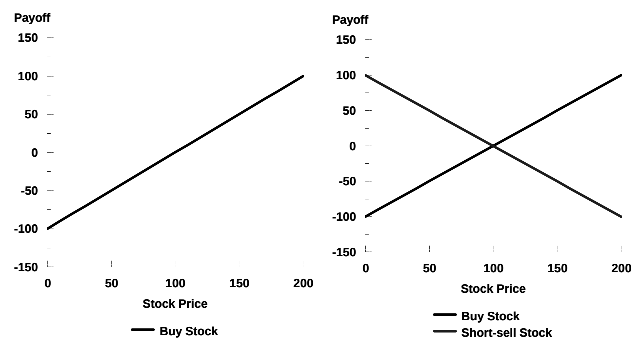
Payoffs to Buying & Selling Put Options [Exercise Price, K=$100]



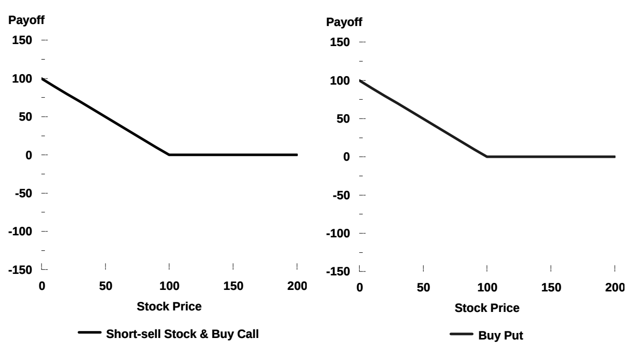
Payoffs to Buying & Selling Straddles [Put & Call Combinations, K=$100]



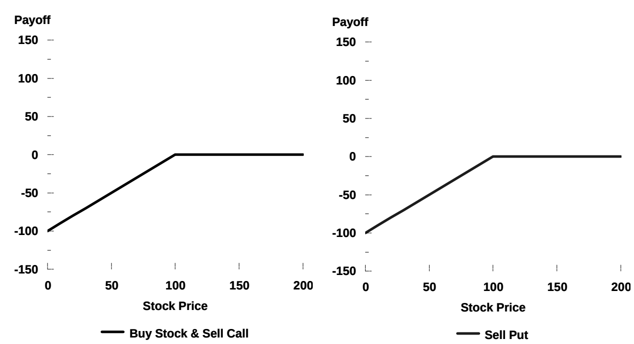
Payoffs to Buying & Short-selling Stock [Purchase Price, K=$100]



Payoffs to Short-selling Stock & Buying a Call vs Buying a Put [K=$100]



Payoffs to Buying Stock & Selling a Call vs Selling a Put [K=$100]



Payoffs Add Up: Useful for Pricing Simple Contingent Claims

Put-call parity is nothing more than the observation that buying a put is equivalent to short-selling the stock & buying a call

- **invest the net proceeds in a riskfree bond**

You can combine basic options with stocks & riskfree bonds to create any payoff structure you like

- **presumably the market will price it "fairly"**
  - **i.e., you will be correctly compensated for the risk you choose to bear**

The Intuition Behind the Black/Scholes Model

- **It is possible to create a portfolio of stocks and bonds that has the exact same payoff as a call option over a very short period of time**
- **Since the stock and bond portfolio and the call option have the exact same payoffs, they must have the same price or there would be pure arbitrage opportunities**
- **Thus, we can value options by identifying this replicating portfolio of stocks and bonds. The stock and bond prices are directly observable**

The Black/Scholes Model A Simple Example

Assume

$S_{T-1} = \$50$

$$S_T = \begin{cases} \$100 \\ \$25 \end{cases}$$

$r = 1.25$

A call option is available with K = \$50.

What is the value of this call option?

The Black/Scholes Model A Simple Example

Consider the following portfolio:

	T-1	T: $S_T = 25$	T: $S_T = 100$
Write 3 calls	3C	0	-150
Buy 2 shares	-100	50	200
Borrow \$40	40	-50	-50
Total	0	0	0

No arbitrage implies 3C - 100 + 40 = 0 or C = \$20

The Black/Scholes Model A Simple Example

We were able to value the call option in this case because we were able to find a stock and bond portfolio (buy 2/3 of a share and borrow \$13.33) that had the exact same payoff as the call option over this one period.

- **If two portfolios have identical payoffs, then the no arbitrage condition implies that they must have the same price.**

Lets try to generalize this reasoning.

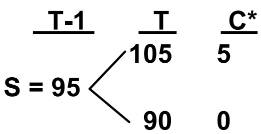
Black/Scholes Model

Suppose the risk-free interest rate is 5%. What is the price of a call option with an exercise price of 100?

T-1	T	C*
S = 95	105	5
	90	0

Create a portfolio of Δ shares of stock and B dollars of bonds where Δ and B are chosen so that the stock and bond portfolio has the exact same payoffs as the call option.

Black/Scholes Model

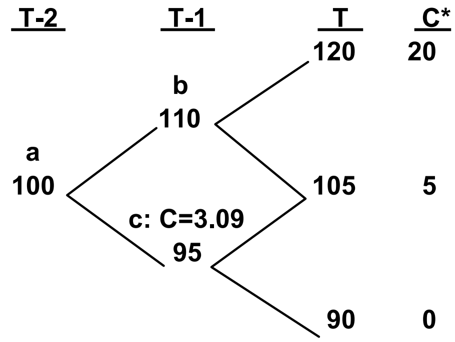


$105\,\Delta + 1.05\,B = 5$
$90\,\Delta + 1.05\,B = 0$
$15\,\Delta = 5;\ \Delta = .3333$

$90(.3333) + 1.05\,B = 0$
$B = -28.57$

$C = S\,\Delta + B$
$= 95(.3333) - 28.57$
$= 3.09$

Black/Scholes Model



Black/Scholes Model

b: $120\,\Delta + 1.05\,B = 20$
$105\,\Delta + 1.05\,B = 5$
$\Delta = 1$

$105 + 1.05\,B = 5$
$B = -95.24$

$C = S\Delta + B$
$= 110\,(1) - 95.24$
$= 14.76$

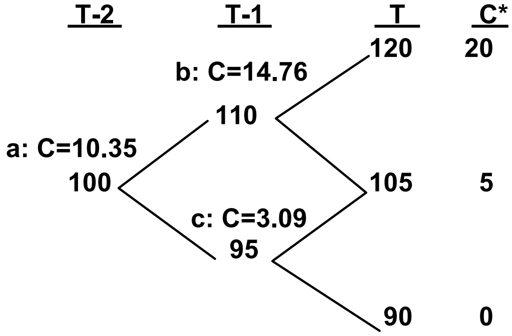
Black/Scholes Model

a: $110\,\Delta + 1.05\,B = 14.76$
$95\,\Delta + 1.05\,B = 3.09$
$\Delta = .7780$

$B = -67.45$

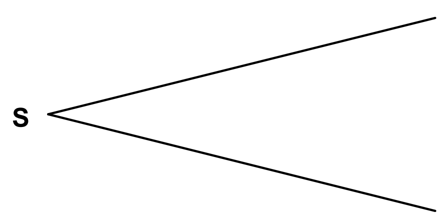
$C = S\Delta + B$
$= 100\,(.7780) - 67.45$
$= 10.35$

Black/Scholes Model



Derivation of The Black/Scholes Model

Consider what happens as you take a fixed interval of time, and divide it into more and more subintervals.



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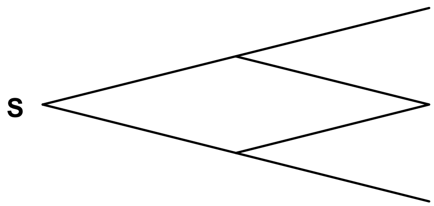

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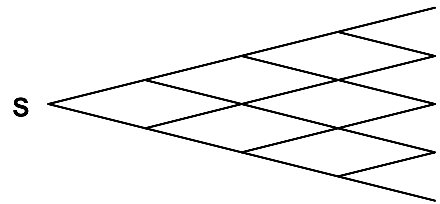

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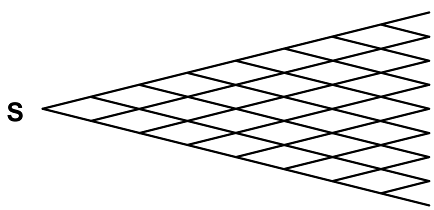

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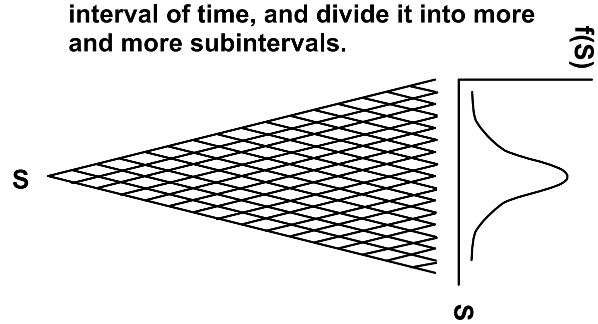


The Black/Scholes Model

Create a hedge portfolio

$$\tilde{V}_H = \tilde{S}\, Q_s + \tilde{C}\, Q_c$$

$$d\tilde{V}_H = d\tilde{S}\, Q_s + d\tilde{C}\, Q_c \qquad (1)$$

So far, this looks like a standard calculus problem.

- The only problem is that C and S are correlated random variables and the standard rules of calculus do not apply.

Ito's Lemma

If C = C(S,t) where C and S are random variables, then

$$dC = (\delta C/\delta S)\, dS + (\delta C/\delta t)\, dt + (1/2)\, (\delta^2 C/\delta S^2)\, \sigma^2 S^2\, dt \qquad (2)$$

Assumptions needed for Ito's lemma:

- Stock prices are continuous
- Stock prices have no memory
- Option price is function of current price but not a function of past price path

The Black/Scholes Model: A Riskfree Hedge

$dV_H = d\tilde{S}\, Q_S + [(\delta C/\delta S)\, d\tilde{S} + (\delta C/\delta t)\, dt$
$\quad + (1/2)\, (d^2C/dS^2)\, \sigma^2 S^2\, dt]\, Q_C$ (3)

Choose Q_S and Q_C so that

$dS\, Q_S + (\delta C/\delta S)\, dS\, Q_C = 0$ (4)

In other words, V_H is risk free as long as

$Q_S/Q_C = -\, (\delta C/\delta S)$

The Black/Scholes Model: A Riskfree Hedge

Since V_H is risk-free, its rate of return must be equal to the risk-free rate, i.e.

$dV_H / V_H = r\, dt.$ (5)

If you make the necessary substitutions of (4) and (5) into (3), you are left with a partial differential equation (without any random variables).

- The solution to this equation [with the boundary condition C* = max (0 , S - K)] is the Black/Scholes Option Pricing Model

The Black/Scholes Model

$$C = S\, N\left\{\frac{\ln(S/K) + (R + \sigma^2/2)T}{\sigma\sqrt{T}}\right\}$$

$$-\exp(-r\,T)\, K\, N\left\{\frac{\ln(S/K) + (R - \sigma^2/2)T}{\sigma\sqrt{T}}\right\}$$

The Black/Scholes Model

Valuing an option with no uncertainty about exercising:

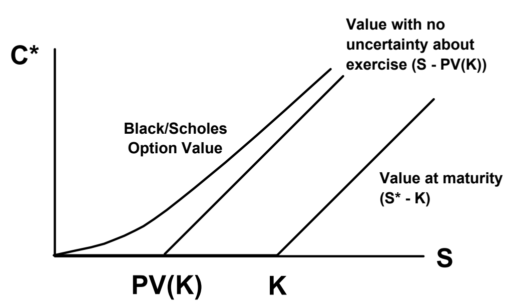
C = PV (C*)

= PV [max (0 , S* - K)]

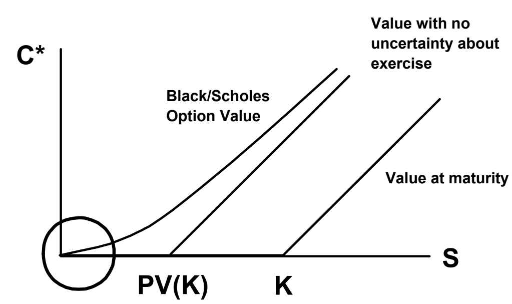
= PV (S* - K) (if option is in the money)

= S - exp(-r T) K

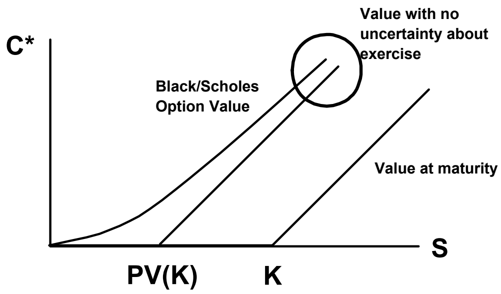
The Black/Scholes Model & Boundary Conditions



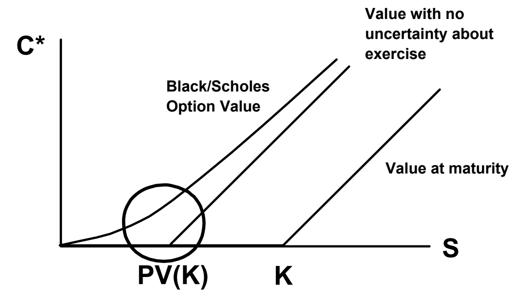
The Black/Scholes Model Out-of-the-Money: S > C > 0



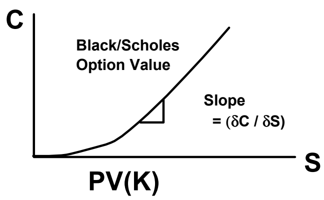
The Black/Scholes Model
In-the-money: C > S - PV(K)



The Black/Scholes Model
At-the-Money Options Are Valuable



The Black/Scholes Riskfree Hedge

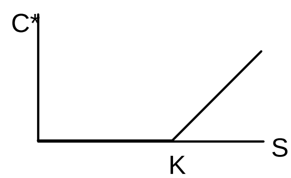


A hedge is risk-free if $Q_S / Q_C = - (\delta C / \delta S)$

On Wall Street, this portfolio is referred to as a Δ hedge

Comparative Statics of the Black/Scholes Model

$$C = C(\overset{+}{S}, \overset{-}{K}, \overset{+}{T}, \sigma^2, r, DIV)$$



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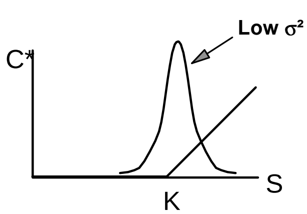

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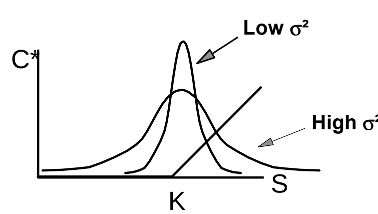
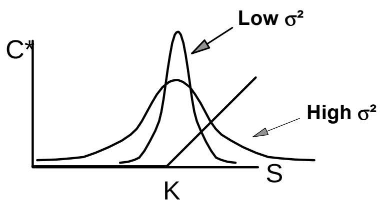


Comparative Statics of the Black/Scholes Model

$$\overset{+}{S},\ \overset{-}{K},\ \overset{+}{T},\ \overset{+}{\sigma^2},\ \overset{+}{r}$$

C = C (S, K, T, σ², r, DIV)



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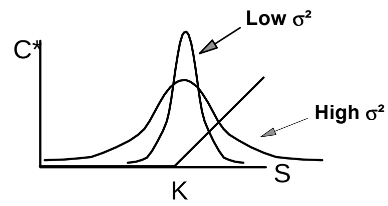

Put-Call Parity

With European options, there is a direct relation between put and call options:

- a put option gives the owner the right to sell stock to the issuer at an exercise price K
- a put is equal to a call, minus the stock plus the discounted exercise price

  P = C - S + K exp(-rT)

- so the Black/Scholes Model can price puts by pricing the call

Valuing American Put Options on Dividend Paying Stocks

- It is not true, in general, that a put option is worth more alive than dead
- The optimal exercise strategy for American put options is more complicated than the optimal exercise strategy for American call options
- The most common time to exercise an American put option is just after an ex-dividend day, but this is not always the case
- There are likely to be larger differences between B/S prices and market prices for puts than for calls

Estimating Volatility Using Options Prices -- Implied Volatility

If you assume that the Black/Scholes model is correct

- you can observe all of the other variables necessary to calculate model prices
- then experiment with different values of volatility σ² until you find one that is consistent with the observed option price
- this is called the implied volatility