FIN 411 -- Investments
Option Pricing

Options: Definitions
A call option gives the buyer the right, but not the obligation,
- to purchase a specific asset
- for a prespecified price
- on a specific future date
American vs. European Options
- exercisable before maturity?

Arbitrage Restrictions on Call Prices

1) $C \geq 0$
Consider the portfolio formed by buying the call option.

<table>
<thead>
<tr>
<th>Today</th>
<th>Future</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-C$</td>
<td>$S^* &lt; K$  $S^* &gt; K$</td>
</tr>
<tr>
<td></td>
<td>$0$      $S^* - K &gt; 0$</td>
</tr>
</tbody>
</table>

2) $C \leq S$
Consider the following portfolio: Buy the stock and sell the call.

<table>
<thead>
<tr>
<th>Today</th>
<th>Future</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C - S$</td>
<td>$S^* \leq K$  $S^* &gt; K$</td>
</tr>
<tr>
<td></td>
<td>$S^<em>$  $S^</em> - (S^* - K) = K$</td>
</tr>
</tbody>
</table>

Arbitrage Restrictions on Call Prices

3) $C \geq S - PV(K)$
Consider the following portfolio: Buy the call, sell the stock, and lend PV(K).

<table>
<thead>
<tr>
<th>Today</th>
<th>Future</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S - C - PV(K)$</td>
<td>$S^* &lt; K$  $S^* &gt; K$</td>
</tr>
<tr>
<td></td>
<td>$K - S^<em>$  $(S^</em> - K) - S^* + K$</td>
</tr>
<tr>
<td></td>
<td>$&gt; 0$     $= 0$</td>
</tr>
</tbody>
</table>
Early Exercise of American Options

Suppose you own a call option and you want to close out your position.

- You can exercise and receive $S - K$
- Or you can sell your option for its current market price $C$
- You choose the alternative that yields the greatest profit (e.g., exercise if $C < S - K$ and sell if $C > S - K$)

Arbitrage Restrictions on American Call Prices

Suppose $C < S - K$ between ex-dividend days. Then buy 1 call, short stock, lend $K$ close out position just before ex-dividend day

<table>
<thead>
<tr>
<th>Today</th>
<th>At Ex-dividend Day</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-C + S - K \geq 0$</td>
<td>$S^* &lt; K$</td>
</tr>
<tr>
<td>$-S^* + (1+r)K &gt; 0$</td>
<td>$S^* &gt; K$</td>
</tr>
</tbody>
</table>

Arbitrage Restrictions on American Call Prices

$C > S - K$ except at expiration or just prior to an ex-dividend day

It is never optimal to exercise an American Call Option except at expiration or possibly just before an ex-dividend day.

A Call Option is "worth more alive than dead"

Arbitrage Restrictions on Put Prices

1) $P \geq 0$
2) $P < K$
3) $P \geq PV(K) - S$
4) $P \geq K - S$
(American Put Only)

Payoffs Diagrams for Contingent Claims

Shows relation between $S$, Payoff and Stock Price for claims with an exercise price, $K = \$100$

- ignores cost of buying the contingent claims/options
- ignores transaction costs
- useful for seeing relations among different contracts

Options: Payoff Diagrams

$C^* = \max [ (S^* - K) , 0 ]$
$C^* = \text{call at maturity } t = T$
$S^* = \text{stock at maturity}$
$K = \text{exercise price}$
Payoffs Add Up: Useful for Pricing Simple Contingent Claims

Put-call parity is nothing more than the observation that buying a put is equivalent to short-selling the stock & buying a call.

- Invest the net proceeds in a riskfree bond

You can combine basic options with stocks & riskfree bonds to create any payoff structure you like.

- Presumably the market will price it "fairly"
  - I.e., you will be correctly compensated for the risk you choose to bear.

The Intuition Behind the Black/Scholes Model

- It is possible to create a portfolio of stocks and bonds that has the exact same payoff as a call option over a very short period of time.

- Since the stock and bond portfolio and the call option have the exact same payoffs, they must have the same price or there would be pure arbitrage opportunities.

- Thus, we can value options by identifying this replicating portfolio of stocks and bonds. The stock and bond prices are directly observable.

The Black/Scholes Model

A Simple Example

Assume:

\[ S_{T-1} = \begin{cases} 
  \$50 \\
  \$100 \\
  \$25 
\end{cases} \]

\[ S_T = \begin{cases} 
  \$100 \\
  \$25 
\end{cases} \]

\[ r = 1.25 \]

A call option is available with \( K = $50 \).

What is the value of this call option?

The Black/Scholes Model

Consider the following portfolio:

\[ \begin{array}{ccc}
  T-1 & T & \\
  \text{Write 3 calls} & 3C & 0 \\
  \text{Buy 2 shares} & -100 & 50 \\
  \text{Borrow $40} & 40 & -50 \\
  \text{Total} & 0 & 0 \\
\end{array} \]

No arbitrage implies \( 3C - 100 + 40 = 0 \)

or \( C = $20 \)

Black/Scholes Model

Suppose the risk-free interest rate is 5%. What is the price of a call option with an exercise price of 100?

\[ \begin{array}{ccc}
  T-1 & T & C^* \\
  S = 95 & 105 & 5 \\
  S = 90 & 90 & 0 \\
\end{array} \]

Create a portfolio of \( \Delta \) shares of stock and \( B \) dollars of bonds where \( \Delta \) and \( B \) are chosen so that the stock and bond portfolio has the exact same payoffs as the call option.

The Black/Scholes Model

A Simple Example

We were able to value the call option in this case because we were able to find a stock and bond portfolio (buy 2/3 of a share and borrow $13.33) that had the exact same payoff as the call option over this one period.

- If two portfolios have identical payoffs, then the no arbitrage condition implies that they must have the same price.

Let's try to generalize this reasoning.
Black/Scholes Model

\[ 105 \Delta + 1.05 B = 5 \]
\[ 90 \Delta + 1.05 B = 0 \]
\[ 15 \Delta = 5; \Delta = 0.3333 \]

\[ S = 95 \]
\[ T - 1 \]
\[ T \]
\[ C^* \]
\[ 105 \]
\[ 5 \]
\[ 90 \]
\[ 0 \]
\[ 90(0.3333) + 1.05 B = 0 \]
\[ B = -28.57 \]
\[ C = S \Delta + B \]
\[ = 95(0.3333) - 28.57 \]
\[ = 3.09 \]

Derivation of The Black/Scholes Model

Consider what happens as you take a fixed interval of time, and divide it into more and more subintervals.
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Derivation of The Black/Scholes Model

Create a hedge portfolio

\[ V_H = S Q_s + C Q_c \]

\[ dV_H = dS Q_s + dC Q_c \]  \hspace{1cm} (1)

So far, this looks like a standard calculus problem.

- The only problem is that \( C \) and \( S \) are correlated random variables and the standard rules of calculus do not apply.

The Black/Scholes Model

Ito’s Lemma

If \( C = C(S,t) \) where \( C \) and \( S \) are random variables, then

\[ dC = \left( \frac{\partial C}{\partial S} \right) dS + \left( \frac{\partial C}{\partial t} \right) dt + \left( \frac{1}{2} \right) \left( \frac{\partial^2 C}{\partial S^2} \right) \sigma^2 S^2 dt \]  \hspace{1cm} (2)

Assumptions needed for Ito’s lemma:
- Stock prices are continuous
- Stock prices have no memory
- Option price is function of current price but not a function of past price path
The Black/Scholes Model: A Riskfree Hedge

\[ dV_H = \tilde{d}S Q_s + [\tilde{\delta}C/\tilde{d}S] \tilde{d}S + (\tilde{\delta}C/\tilde{d}t) dt + (1/2) (d^2C/dS^2) \sigma^2 S^2 dt \] \( Q_c \) \hspace{1cm} (3)

Choose \( Q_s \) and \( Q_c \) so that

\[ dS Q_s + (\delta C/\delta S) dS Q_c = 0 \] \hspace{1cm} (4)

In other words, \( V_H \) is risk free as long as

\[ Q_s/Q_c = - (\delta C/\delta S) \]

The Black/Scholes Model

\[ C = S N \left( \frac{\ln(S/K) + (R + \sigma^2/2)T}{\sigma \sqrt{T}} \right) - \exp(-rT) K N \left( \frac{\ln(S/K) + (R - \sigma^2/2)T}{\sigma \sqrt{T}} \right) \]

Valuing an option with no uncertainty about exercising:

\[ C = PV(C^*) \]
\[ = PV [ \max(0, S^* - K) ] \]
\[ = PV(S^* - K) \] (if option is in the money)
\[ = S - \exp(-rT) K \]

The Black/Scholes Model & Boundary Conditions

Out-of-the-Money: \( S > C > 0 \)

Value at maturity \( (S^* - K) \)

Value with no uncertainty about exercise \( (S - PV(K)) \)

Black/Scholes Option Value

PV(K) K S
The Black/Scholes Model

**In-the-money:** \[ C > S - PV(K) \]

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**At-the-Money Options Are Valuable**

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**The Black/Scholes Riskfree Hedge**

A hedge is risk-free if \( \frac{Q_s}{Q_C} = -\frac{\delta C}{\delta S} \)

On Wall Street, this portfolio is referred to as a \( \Delta \) hedge.

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**Comparative Statics of the Black/Scholes Model**

\[ C = C(S, K, T, \sigma^2, r, \text{DIV}) \]

- \( \sigma^2 \) Low
- \( \sigma^2 \) High

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Put-Call Parity

With European options, there is a direct relation between put and call options:
- a put option gives the owner the right to sell stock to the issuer at an exercise price \( K \)
- a put is equal to a call, minus the stock plus the discounted exercise price

\[ P = C - S + K \exp(-rT) \]

so the Black/Scholes Model can price puts by pricing the call

Valuing American Put Options on Dividend Paying Stocks

- It is not true, in general, that a put option is worth more alive than dead
- The optimal exercise strategy for American put options is more complicated than the optimal exercise strategy for American call options
- The most common time to exercise an American put option is just after an ex-dividend day, but this is not always the case
- There are likely to be larger differences between B/S prices and market prices for puts than for calls

Estimating Volatility Using Options Prices -- Implied Volatility

If you assume that the Black/Scholes model is correct
- you can observe all of the other variables necessary to calculate model prices
- then experiment with different values of volatility \( \sigma^2 \) until you find one that is consistent with the observed option price

- this is called the implied volatility